**Module 6: Sequences and Series**

**Topics**

1. [Sequences and Series](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#I)
   1. [Sequences](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#Ia)
   2. [Recursion Formulas](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#Ib)
   3. [Factorial Notation](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#Ic)
   4. [Summation Notation (Series)](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#Id)
2. [Arithmetic Sequences](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#II)
   1. [Arithmetic Sequences](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#IIa)
   2. [General Term of an Arithmetic Sequence](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#IIb)
   3. [Sum of the First *n* Terms of an Arithmetic Sequence](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#IIc)
3. [Geometric Sequences](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#III)
   1. [Geometric Sequences](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html#IIIa)
   2. [General Term of a Geometric Sequence](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html" \l "IIIb)
   3. [Sum of the First](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html" \l "IIIc)*[n](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html" \l "IIIc)*[Terms of a Geometric Sequence](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/S3-Commentary.html" \l "IIIc)

**I. Sequences and Series**

After completing this section, you should be able to:

* find particular terms of a sequence from the general term
* use recursion formulas to solve problems
* use factorial notation
* use summation notation

**A. Sequences**

Many creations in nature involve intricate mathematical designs, including a variety of spirals. For example, the individual florets in the head of a sunflower are arranged in spirals. In some species, there are 21 spirals in the clockwise direction and 34 in the counterclockwise direction. The precise numbers depend on the species of the sunflower: 21 and 34, or 34 and 55, or 55 and 89, or even 89 and 144.

This observation becomes even more interesting upon consideration of a sequence of numbers investigated by a thirteenth-century Italian mathematician named Leonardo of Pisa, also known as Fibonacci. The **Fibonacci sequence** of numbers is an infinite sequence that begins as follows:

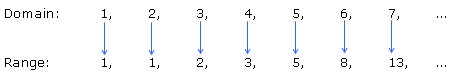
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

The first two terms are 1. Every term thereafter is the sum of the two preceding terms. For example, the third term, 2, is the sum of the first and second terms: 1 + 1 = 2. The fourth term, 3, is the sum of the second and third terms: 1 + 2 = 3, and so on. Did you know that the number of spirals in a daisy or a sunflower, 21 and 34, are two Fibonacci numbers? The number of spirals in a pine cone, 8 and 13, and in a pineapple, again 8 and 13, are also Fibonacci numbers.

The Fibonacci sequence can be thought of as a function. The terms of the sequence

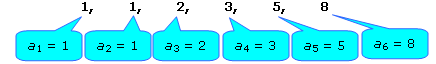
           1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

are the range values for a function whose domain is the set of positive integers.



Thus, *f*(1) = 1, *f*(2) = 1, *f*(3) = 2, *f*(4) = 3, *f*(5) = 5, *f*(6) = 8, *f*(7) = 13, and so on.

The letter *a* with a subscript numeral, rather than the usual function notation, is used to represent function values of a sequence. The subscripts make up the domain of the sequence and identify the location of a term. Thus, *a*1 represents the first term of the sequence, *a*2represents the second term, *a*3 represents the third term, and so on. In this notation, the first six terms of the Fibonacci sequence are:



The notation *an* represents the *n*th term, or **general term**, of a sequence. The entire sequence is represented by {*an*}.

**Definition of a Sequence**

An **infinite sequence** {*an*} is a function whose domain is the set of positive integers. The function values, or **terms**, of the sequence are represented by

*a*1, *a*2, *a*3, *a*4, …, *an*, … .

Sequences whose domains consist only of the first *n* positive integers are called **finite sequences**.

**Example I.A.1:** Write the first four terms of the sequence whose *n*th term is:

a. *an* = 3*n* + 4

b. *an* =https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1a.gif

**Solution:**

a. To find the first four terms of the sequence whose general term is *an* = 3*n* + 4, replace ***n*** in the formula with **1**, **2**, **3**, and **4**:

|  |  |
| --- | --- |
| Substitute *n* = **1**: | *a*1 = 3(**1**) + 4 = 3 + 4 = 7 |
| Substitute *n* = **2**: | *a*2 = 3(**2**) + 4 = 6 + 4 = 10 |
| Substitute *n* = **3**: | *a*3 = 3(**3**) + 4 = 9 + 4 = 13 |
| Substitute *n* = **4**: | *a*4 = 3(**4**) + 4 = 12 + 4 = 16 |

The first four terms are 7, 10, 13, and 16. The sequence defined by *an* = 3*n* + 4 can be written as

7, 10, 13, 16, …, 3*n* + 4, … .

b. To find the first four terms of the sequence whose general term is *an* = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1a.gif,   
replace ***n*** in the formula with **1**, **2**, **3**, and **4**:

Substitute *n* = **1**:  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1c.png

Substitute *n* = **2**: https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1d.png

Substitute *n* = **3**: https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1e.gif

Substitute *n* = **4**:  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1f.gif

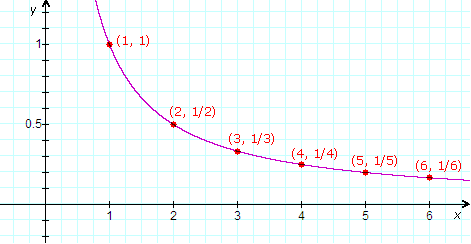
The first four terms are  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1g.gif.

The sequence defined by *an* = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1a.gif

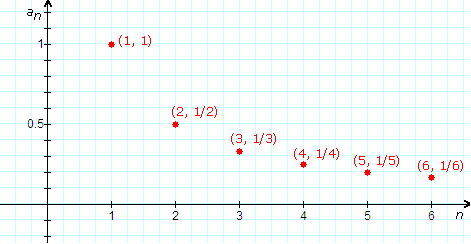
can be written as

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/ExampleVI-A-1i.gif

Because a sequence is a function whose domain is the set of positive integers, the graph of a sequence is a set of discrete points. For example, consider the sequence whose general form is *an* = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/1-ovr-n.gif. How does the graph of this sequence differ from the graph of the function*f*(*x*) = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/1-ovr-x.gif? The graph of *f*(*x*) = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/1-ovr-x.gif for positive values of *x* is shown below:



To obtain the graph of the sequence {*an*} = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/1-ovr-n-brac.gif, remove all points from the graph except for those whose *x*-coordinates are positive integers. Thus, remove all points except (1, 1), (2, https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/one-half.gif), (3, https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/one-third.gif), (4, https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/one-quartr.gif), and so on. The remaining points are the graph of the sequence {*an*} = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/1-ovr-n-brac.gif, shown below.



Note another difference between the graph of a sequence and the graph of a function: here, the horizontal axis is labeled *n*, and the vertical axis is labeled *an*.

**B. Recursion Formulas**

In the first example, the formulas given for the *n*th term of a sequence expressed the term as a function of *n*, the number of the term. Sequences can also be defined using **recursion formulas**. A recursion formula defines the *n*th term of a sequence as a function of the previous terms. The next example illustrates that if the first term of a sequence is known, then the recursion formula can be used to determine the remaining terms.

**Example I.B.1:** Find the first four terms of the sequence in which *a*1 = 5 and *an* = 3*an*– 1 + 2 for *n* &ge; 2.

**Solution:**

|  |  |
| --- | --- |
| *a*1 = 5 | This is the given first term. |
| *a*2 = 3*a*1 + 2 | Use *an* = 3*an*– 1 + 2, with *n* = 2. Thus, *a*2 = 3*a*2 – 1 + 2 = 3*a*1 + 2. |
| = 3(5) + 2 = 17 | Substitute 5 for *a*1. |
| *a*3 = 3*a*2 + 2 | Again use *an* = 3*an*– 1 + 2, with *n* = 3. |
| = 3(17) + 2 = 53 | Substitute 17 for *a*2. |
| *a*4 = 3*a*3 + 2 | Notice *a*4 is defined in terms of *a*3. |
| = 3(53) + 2 = 161 | Substitute 53 for *a*3. |

The first four terms are 5, 17, 53, and 161.

**C. Factorial Notation**

Products of consecutive positive integers occur quite often in sequences. These products can be expressed in a special notation called **factorial notation**.

**Factorial Notation**

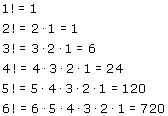
If *n* is a positive integer, the notation ***n*!** (read "*n* factorial") is the product of all the positive integers from n down through 1.

*n*! = *n*(*n* – 1)(*n* – 2) … (3)(2)(1)

0! (zero factorial) by definition is 1.

0! = 1

The values of *n*! for the first six positive integers are



A factorial symbol applies only to the number or variable that it follows unless grouping symbols appear. For example,

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/fact-notation1.gif

whereas

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/fact-notation2.gif

In this sense, factorials are similar to exponents.

**Example I.C.1:** Write the first four terms of the sequence whose *n*th term is

*an* =https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-1.gif

**Solution:**

To find the first four terms of the sequence whose general term is *an* =https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-1.gif, replace each ***n*** in https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-1.gif with **1**, **2**, **3**, and **4**.

|  |  |
| --- | --- |
| Substitute *n* = **1**: | *a*1 = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-I-C-1a-soltn.gif     (0! = 1, by definition) |
| Substitute *n* = **2**: | *a*2 = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-I-C-1b-soltn.png |
| Substitute *n* = **3**: | *a*3 = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-I-C-1c-soltn.png |
| Substitute *n* = **4**: | *a*4 = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-I-C-1d-soltn.png |

The first four terms are 2, 4, 4, and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/8-ovr-3.gif.

The sequence defined by *an* =https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-1.gif can be

written as

2, 4, 4, https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/8-ovr-3.gif, … https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-1.gif, … .

**Example I.C.2:**Evaluate each factorial expression:

|  |  |
| --- | --- |
| a. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-2a.gif | b.https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-2b.gif |

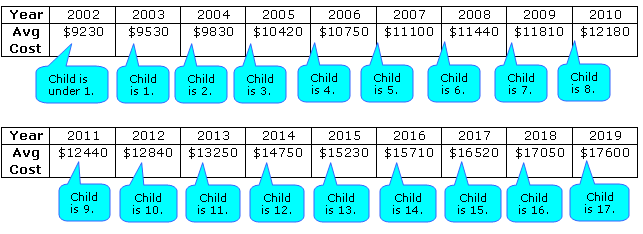
**Solution:**

a. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-2a.gif= https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-I-C-2a-soltn.png= https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/90-ovr-2.gif= 45

b. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-c-2b.gif= https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-I-C-2b-soltn.png= *n* + 1

**D. Summation Notation (Series)**

It is sometimes useful to find the sum of the first *n* terms of a sequence. For example, consider the annual cost of raising a child born in the United States in 2002 to a middle income ($39,700–$66,900) family, shown in the following table:



Let *an* represent the cost of raising a child in year *n*, where *n* = 1 corresponds to 2002, *n* = 2 to 2003, *n* = 3 to 2004, and so on. The terms of the finite sequence are given as follows:

|  |  |  |  |
| --- | --- | --- | --- |
| *a*1 = 9,230 | *a*6 = 11,100 | *a*11 = 12,840 | *a*16 = 16,520 |
| *a*2 = 9,530 | *a*7 = 11,440 | *a*12 = 13,250 | *a*17 = 17,050 |
| *a*3 = 9,830 | *a*8 = 11,810 | *a*13 = 14,750 | *a*18 = 17,600 |
| *a*4 = 10,420 | *a*9 = 12,180 | *a*14 = 15,230 |  |
| *a*5 = 10,750 | *a*10 = 12,440 | *a*15 = 15,710 |  |

What information would be described if the terms of this sequence were added? Adding the terms of this sequence gives the total cost of raising a child born in 2002 from birth through age 17. Thus,

*a*1 + *a*2 + *a*3 + *a*4 + *a*5 + *a*6 + *a*7 + *a*8 + *a*9 + *a*10 + *a*11 + *a*12 + *a*13 + *a*14 + *a*15 + *a*16 + *a*17 + *a*18  
= 9,230 + 9,530 + 9,830 + 10,420 + 10,750 + 11,100 + 11,440 + 11,810 + 12,180 + 12,440 + 12,840 + 13,250 + 14,750 + 15,230 + 15,710 + 16,520 + 17,050 + 17,600  
= 231,680.

The total cost of raising a child born in 2002 from birth through age 17 is $231,680.

There is a compact notation for expressing the sum of the first *n* terms of a sequence. Rather than write  
*a*1 + *a*2 + *a*3 + *a*4 + *a*5 + *a*6 + *a*7 + *a*8 + *a*9 + *a*10 + *a*11 + *a*12 + *a*13 + *a*14 + *a*15 + *a*16 + *a*17 + *a*18  
we can use **summation notation** to express the sum as

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/summation-notation.gif

Read this expression as "the sum as *k* goes from 1 to 18 of *ak*." The letter *k* is called the **index of summation**. The symbol ∑ (the uppercase Greek letter sigma) instructs us to add all of the terms of the sequence.

**Summation Notation**

The sum of the first *n* terms of a sequence is represented by the **summation notation**

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/summation1.gif= *a*1 + *a*2 + *a*3 + *a*4 + … + *an*

where *k* is the **index of summation**, *n* is the **upper limit of summation**, and 1 is the **lower limit of summation**.

Any letter can be used for the index of summation. The letters *i*, *j*, and *k* are used commonly. Furthermore, the lower limit of summation can be a positive integer other than 1.

When a sum expressed in summation notation is written out, the sum is said to be written in **expanded form**.

**Example I.D.1:** Expand and evaluate the sums:

|  |  |  |
| --- | --- | --- |
| a. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1a.gif | b. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1b.gif | c. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1c.gif |

**Solution:**

a. Replace *k* in the expression *k*2 + 1 with each integer from 1 through 6, then add.  
https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1a.gif = (**1**2 + 1) + (**2**2 + 1) + (**3**2 + 1) + (**4**2 + 1) + (**5**2 + 1) + (**6**2 + 1)  
                = 2 + 5 + 10 + 17 + 26 + 37  
                = 97

b. The index of summation in https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1b.gif is *j*. First, evaluate (–2)*j* – 5 for each integer from 4 through 7, then add.

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1b.gif = [(–2)**4** – 5] + [(–2)**5** – 5] + [(–2)**6** – 5] + [(–2)**7** – 5]  
                    = (16 – 5) + (–32 – 5) + (64 – 5) + (–128 – 5)  
                    = 11 + (–37) + 59 + (–133)  
                    = –100

c. Notice that every term of the sum is 3. The notation *k* = 1 through 5 indicates adding the first five terms of a sequence in which every term is 3.  
https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-d-1c.gif = 3 + 3 + 3 + 3 + 3 = 15

**Example I.D.2:** Express each sum using summation notation:

a. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-D-2a.gif        b. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-D-2b.gif

**Solution:**

In each case, 1 will be used as the lower limit of summation and *k* will be used for the index of summation.

a. The sum 13 + 23 + 33 + … + 73 has seven terms, each of the form *k*3, starting at *k* = 1 and ending at *k* = 7. Thus,

13 + 23 + 33 + … + 73 = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-D-2a-soltn.gif

b. The sum https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-D-2b.gif has *n* terms, each of the form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/1-ovr-3n-1.gif, starting at *k* = 1 and ending at *k* = *n*. Thus,

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-D-2b.gif = https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example1-D-2bb-soltn.gif

The following table contains some important properties of sums expressed in summation notation.

|  |  |
| --- | --- |
| **Summation Properties** | **Example** |
| 1. https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties1.gif,where *c* is any real number | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex1.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex1a.gif= 3(12 + 22 + 32 + 42)  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/graphs/mod6-secI-summatn-propties-ex1bb.gif  **Conclusion:** https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex1c.gif |
| 2.https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties2.gif | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex2.gif  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex2a.gif                    https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/graphs/mod6-secI-summatn-propties-ex2aa.gif **Conclusion:** https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex2b.gif |
| 3.https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties3.gif | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/mod6-secI-summatn-propties-ex3.gif                     = (32 – 33) + (42 – 43) + (52 – 53) **Conclusion:** |

**II. Arithmetic Sequences**

After completing this section, you should be able to:

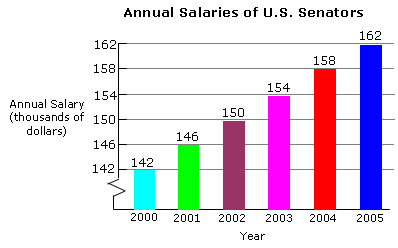
* find the common difference of an arithmetic sequence
* write the terms of an arithmetic sequence
* use the formula for the general term of an arithmetic sequence to solve problems
* use the formula for the sum of the first *n* terms of an arithmetic sequence

**A. Arithmetic Sequences**

The bar graph below shows approximated annual salaries, rounded to the nearest thousand dollars, of U.S. senators from 2000 to 2005. The graph illustrates that each year, salaries increased by $4,000. The sequence of annual salaries (in thousands)

142, 146, 150, 154, 158, 162, …

shows that each term after the first term, 142, differs from the preceding term by a constant amount, namely, 4. This sequence is an example of an **arithmetic sequence**.



(Source: U.S. Congressional Research Service)

**Definition of an Arithmetic Sequence**

An **arithmetic sequence** is a sequence in which each term after the first differs from the preceding term by a constant amount. The difference between consecutive terms is called the **common difference** of the sequence.

The common difference, *d*, is found by subtracting any term from the term that directly follows it. In the following examples, the common difference is found by subtracting the first term from the second term, *a*2 – *a*1.

|  |  |
| --- | --- |
| **Arithmetic Sequence**  142, 146, 150, 154, 158, … –5, –2, 1, 4, 7, … 8, 3, –2, –7, –12, … | **Common Difference**  *d* = 146 – 142 = 4 *d* = –2 – (–5) = –2 + 5 = 3 *d* = 3 – 8 = –5 |

The graphs of the arithmetic sequences –5, –2, 1, 4, 7, … and 8, 3, –2, –7, –12, … are shown below.

|  |  |
| --- | --- |
| Sequence {*an*} = –5, –2, 1, 4, 7, … | Sequence {*bn*} = 8, 3, –2, –7, –12, … |

The graph of each arithmetic sequence forms a set of discrete points lying on a straight line. This illustrates that an **arithmetic sequence** is a linear function whose domain is the set of positive integers.

If the first term of an arithmetic sequence is *a*1, each term after the first is obtained by adding *d*, the common difference, to the previous term. This can be expressed recursively as follows:

*an* = *an* – 1 + *d*

Add *d* to the term in any position to obtain the next term. To use this recursion formula, the first term must be known.

**Example II.A.1:** In 1980, 77.4 percent of U.S. men were working in the labor force. Using the data in the graphs and text below, find the first five terms of this arithmetic sequence, in which *a*1 = 77.4 and *an* = *an*– 1 – 0.67:

|  |
| --- |
|  |

|  |
| --- |
| (Source: U.S. Department of Labor) |

The charts show the percentage of men and women in the U.S. labor force for five-year periods starting with 1980. The recursion formula *an* = *an*– 1 – 0.67 models the percentage of men working in the U.S. labor force, *an*, for each five-year period starting with 1980. Thus, *n* = 1 corresponds to 1980, *n* = 2 to 1985, *n* = 3 to 1990, and so on.

**Solution:**

The recursion formula *a*1 = 77.4 and *an* = *an*– 1 – 0.67 indicates that each term after the first is obtained by adding –0.67 to the previous term. Thus, during each five-year period, the percentage of men in the labor force decreased by 0.67 percent.

|  |  |
| --- | --- |
| *a*1 = 77.4 |  |
| *a*2 = *a*1 – 0.67 = 77.4 – 0.67 = 76.73 | Use *an* = *an*– 1 – 0.67 with *n* = 2 |
| *a*3 = *a*2 – 0.67 = 76.73 – 0.67 = 76.06 | Use *an* = *an*– 1 – 0.67 with *n* = 3 |
| *a*4 = *a*3 – 0.67 = 76.06 – 0.67 = 75.39 | Use *an* = *an*– 1 – 0.67 with *n* = 4 |
| *a*5 = *a*4 – 0.67 = 75.39 – 0.67 = 74.72 | Use *an* = *an*– 1 – 0.67 with *n* = 5 |

The first five terms are 77.4, 76.73, 76.06, 75.39, and 74.72.

These numbers represent the percentage of men working in the U.S. labor force in 1980, 1985, 1990, 1995, and 2000, respectively, as given by the model.

**B. General Term of an Arithmetic Sequence**

Consider an arithmetic sequence whose first term is *a*1 and whose common difference is *d*. To find a formula for the general term, *an*, of an arithmetic sequence, begin by writing the first six terms. The first term is *a*1. The second term is *a*1 + *d*. The third term is *a*1 + *d* + *d*, or *a*1 + 2*d*. Thus, starting with *a*1 and adding *d* to each successive term, the first six terms are:

*a*1, *a*1 + *d*, *a*1 + 2*d*, *a*1 + 3*d*, *a*1 + 4*d*, *a*1 + 5*d*.

Compare the coefficient of *d* and the subscript of a denoting the term number. The coefficient of *d* is one less than the subscript of a denoting the term number.

|  |  |
| --- | --- |
| *a*3: third term = *a*1 + 2*d* | *a*4: fourth term = *a*1 + 3*d* |

Thus, the formula for the *n*th term is

*an*: *n*th term = *a*1 + (*n* – 1)*d  
*

**General Term of an Arithmetic Sequence**

The *n*th term (general term) of an arithmetic sequence with the first term *a*1 and common difference *d* is

*an* = *a*1 + (*n* – 1)*d*

**Example II.B.1:** Find the eighth term of the arithmetic sequence whose first term is 4 and whose common difference is –7.

**Solution:**

To find the eighth term, *a*8, replace *n* in the formula with 8, *a*1 with 4, and *d* with –7.

*an* = *a*1 + (*n* – 1)*d*  
*a*8 = 4 + (8 – 1)(–7) = 4 + 7(–7) = 4 + (–49) = –45

The eighth term is –45. To check this result, write the first eight terms of the sequence:

4, –3, –10, –17, –24, –31, –38, –45

**C. Sum of the First *n* Terms of an Arithmetic Sequence**

The sum of the first *n* terms of an arithmetic sequence, denoted *Sn* and called the ***n*th partial sum**, can be found without having to add up all the terms. Let

*Sn* = *a*1 + *a*2 + *a*3 + … + *an*

be the sum of the first *n* terms of an arithmetic sequence. Because *d* is the common difference between terms, *Sn* can be written forward and backward as follows.

*Sn*  =   *a*1 +  (*a*1 + d) +  (*a*1 + 2d) + …  + *an*  
*Sn*  =   *an* +  (*an* – d) +  (*an* – 2d) + …   + *a*1  
\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_  
2*Sn* = (*a*1 + *an*) + (*a*1 + *an*)  + (*a*1 + *an*) + …   + (*a*1 + *an*)       Add the two equations.

Because there are *n* sums of (*a*1 + *an*) on the right side, express this side as *n*(*a*1 + *an*).  
Thus, the last equation can be written as follows:

|  |  |
| --- | --- |
| 2*Sn* = *n*(*a*1 + *an*) |  |
| *Sn* = (*a*1 + *an*) | Solve for *Sn* by dividing both sides by 2. |

This proves the following result:

**The Sum of the First *n* Terms of an Arithmetic Sequence**

The sum *Sn* of the first *n* terms of an arithmetic sequence is given by

*Sn* = (*a*1 + *an*)

where *a*1 is the first term and *an* is the *n*th term.

To find the sum of the first 100 terms of the arithmetic sequence using *Sn* = (*a*1 + *an*), the first term, *a*1, must be known; the last term, *an*, must be known; and the number of terms, *n*, must also be known.

**Example II.C.1:** Find the sum of the first 100 terms of the arithmetic sequence: 1, 3, 5, 7, …

**Solution:**

To find the sum of the first 100 terms, *S*100, replace *n* in the formula with 100.

*Sn* = (*a*1 + a*n*)  
*S*100 = (*a*1 + a100)  
*S*100 = (1 + a100)

Use the formula for the general term of a sequence to find *a*100. The common difference, *d*, of 1, 3, 5, 7, …, is 2.

|  |  |
| --- | --- |
| *an* = *a*1 + (*n* – 1)*d* | This is the formula for the *n*th term of an arithmetic sequence. Use this information to find the 100th term. |
| *a*100 = 1 + (100 – 1)(2) | Substitute 100 for *n*, 2 for *d*, and 1 (the first term) for *a*1. |
| *a*100 = 1 + (99)(2) |  |
| *a*100 = 199 |  |

Now find the sum of the 100 terms 1, 3, 5, 7, …, 199.

*Sn* = (*a*1 + *an*)  
*S*100 = (1 + 199) = 50(200) = 10,000

The sum of the first 100 odd numbers is 10,000. Equivalently, the 100th partial sum of the sequence 1, 3, 5, 7, … is 10,000.

**Example II.C.2:** Find the following sum: .

**Solution:**

= (5 ∙ 1 − 9) + (5 ∙ 2 − 9) + (5 ∙ 3 − 9) + … + (5 ∙ 25 − 9)  
               =     –4        +      1        +       6       + … +      116

By evaluating the first three terms and the last term, it can be seen that *a*1 = –4; *d*, the common difference, is 1 – (– last term, is 116.

|  |  |
| --- | --- |
| *Sn* = (*a*1 + *an*) | Use the formula for the sum of the first *n* terms of an arithmetic sequence. Let *n* = 25, *a*1 = –4, and *a*25, the last term, is 116. |
| *S*25 = (–4 + 116) = (112) = 1,400 | |

Thus,

 = 1,400

**III. Geometric Sequences**

After completing this section, you should be able to:

* find the common ratio of a geometric sequence
* write terms of a geometric sequence
* use the formula for the general term of a geometric sequence to solve problems
* use the formula for the sum of the first *n* terms of a geometric sequence

**A. Geometric Sequences**

You are at the closing moments of a job interview, shaking hands with the manager. You've answered all the tough questions without losing your poise, and now you've been offered a job. As a matter of fact, your qualifications are so strong that you've been offered two jobs—one just the day before, with a rival company in the same field. One company offers $30,000 the first year, with increases of six percent per year for four years after that. The other offers $32,000 the first year, with annual increases of three percent per year after that. Over a five-year period, which is the better offer?

If a salary raise is a certain percent each year, the yearly salaries over time form a geometric sequence. After completing this section, you will be able to decide which job offer to accept: you will know which company will pay you more over the course of the next five years.

**Definition of a Geometric Sequence**

A **geometric sequence** is a sequence in which each term after the first is obtained by *multiplying* the preceding term by a fixed nonzero constant. The amount by which each term is multiplied is called the **common ratio** of the sequence.

The common ratio, *r*, is found by dividing any term after the first term by the term that directly precedes it. In the following examples, the common ratio is found by dividing the second term by the first term, .

|  |  |
| --- | --- |
| **Geometric Sequence** | **Common Ratio** |
| 1, 5, 25, 125, 625, … |  |
| 4, 8, 16, 32, 64, … |  |
| 6, –12, 24, –48, 96, … |  |
| 9, –3, 1, – , , … |  |

**Example III.A.1:** Write the first six terms of the geometric sequence with the first term 6 and common ratio .

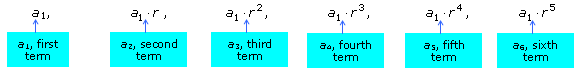
**Solution:**

The first term is 6. The second term is , or 2. The third term is , or . The fourth term is , or , and so on. The first six terms are

6, 2, , , , and .

**B. General Term of a Geometric Sequence**

Consider a geometric sequence whose first term is *a*1 and whose common ratio is *r*. To find a formula for the general term *an* of a geometric sequence, begin by writing the first six terms. The first term is *a*1. The second term is . The third term is , or . The fourth term is , or , and so on. Starting with *a*1 and multiplying each successive term by *r*, the first six terms are



Compare the exponent of *r* to the subscript of *a*, which denotes the term number. Note that the exponent of *r* is one less than the subscript of *a*.



Thus, the formula for the *n*th term is

*an* = *a*1*rn*– 1  


**General Term of a Geometric Sequence**

The *n*th term (general term) of a geometric sequence with first term *a*1 and common ratio *r* is

*an* = *a*1*rn* – 1

**Example III.B.1:** Find the eighth term of the geometric sequence whose first term is –4 and whose common ratio is –2.

**Solution:**

To find the eighth term, *a*8, replace the *n* in the formula with 8, *a*1 with –4, and *r* with –2.

*an* = *a*1*rn*– 1  
*a*8 = –4(–2)8 – 1 = –4(–2)7 = –4(–128) = 512

The eighth term is 512. Check this result by writing the first eight terms of the geometric sequence:

–4, 8, –16, 32, –64, 128, –256, 512

*College Algebra* included the study of exponential functions of the form *f*(*x*) = *bx* and the explosive exponential growth of world population. In the example below, consider a representation of Florida's geometric population, in which the domain is the set of positive integers. Geometric and exponential growth are the same thing.

**Example III.B.2:** The population of Florida from 1990 through 1997 is shown in the following table:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Year** | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 |
| **Population (millions)** | 12.94 | 13.20 | 13.46 | 13.73 | 14.00 | 14.28 | 14.57 | 14.86 |

(Source: U.S. Census Bureau)

a. Show that the population is increasing geometrically.

b. Write the general term for the geometric sequence describing population growth for Florida *n* years after 1989.

c. Estimate Florida's population, in millions, for the year 2000.

**Solution:**

a. First, divide the population for each year by the population in the preceding year.

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/Example-III-B-3a.png

Continuing in this manner, the ratio will remain at approximately 1.02. This means the population is increasing geometrically, with *r*≈ 1.02. In this situation, the common ratio is the growth rate, indicating that the population of Florida in any year shown in the table is approximately 1.02 times the population the year before.

b. The sequence of Florida's population growth is

12.94, 13.20, 13.46, 13.73, 14.00, 14.28, 14.57, 14.86, …

Because the population is increasing geometrically, the general term of the sequence can be determined using

*an* = *a*1*rn*– 1

In this sequence, *a*1 = 12.94, and we know from part (a), *r* ≈ 1.02. Substituting these values into the formula for the general term gives

*an* = 12.94(1.02)*n*– 1

c. Use the formula for the general term, *an*, that you determined in part (b) to estimate Florida's population for the year 2000. The year 2000 is 11 years after 1989, and 2000 – 1989 =&nbsp11, so *n* = 11. Substituting 11 for *n* in *an* = 12.94(1.02)*n*– 1 gives:

a11 = 12.94(1.02)11 – 1 = 12.94(1.02)10 ≈ 15.77

The formula indicates that Florida had a population of approximately 15.77 million in the year 2000. According to the U.S. Census Bureau, Florida's population in 2000 was 15.98 million. This geometric sequence models actual population growth reasonably well.

**C. Sum of the First *n* Terms of a Geometric Sequence**

The sum of the first *n* terms of the geometric sequence, denoted *Sn* and called the ***n*th partial sum**, can be found without having to add up all the terms. Recall that the first *n* terms of a geometric sequence are

*a*1, *a*1*r*, *a*1*r*2, *a*1*r*3, … , *a*1*rn* – 1

To find the formula for the *n*th partial sum, proceed as follows:

|  |  |
| --- | --- |
| (1) *Sn* = *a*1 + *a*1*r* + *a*1*r*2 + *a*1*r*3 + … + *a*1*rn* – 1 | *Sn* is the sum of the first *n* terms of the sequence. |
| (2) *rSn* = *a*1*r* + *a*1*r*2 + *a*1*r*3 + *a*1*r*4 + … + *a*1*rn \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_* | Multiply both sides by *r*. |
| *Sn* – *rSn* = *a*1 – *a*1*rn* | Subtract the second equation from the first equation. |
| *Sn*(1 – *r*) = *a*1(1 – *rn*) | Factor out *Sn* on the left and *a*1 on the right. |
| *Sn* =https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M6-Module_6/images/geometric-seq.gif | Solve for *Sn* by dividing both sides by 1 – *r* (assuming *r* ≠ 1). |

This proves the following result:

**The Sum of the First *n* Terms of a Geometric Sequence**

The sum, *Sn*, of the first *n* terms of a geometric sequence is given by

*Sn* = 

where *a*1 is the first term and *r* is the common ratio (*r* ≠ 1).

To find the sum of the terms of a geometric sequence, the following must be known: first term, *a*1; the common ratio, *r*; and the number of terms, *n*. The following example illustrates how use this formula.

**Example III.C.1:** Find the sum of the first 18 terms of the geometric sequence: 2, –8, 32, –128, ….

**Solution:** To find the sum the first 18 terms, *S*18, replace *n* in the formula with 18.

*Sn* = ,

*S*18 = , *a*1 = 2; *r* must be found.

Find the common ratio, *r*, by dividing the second term of 2, –8, 32, 128, … by the first term.

*r* =  = – = –4

Now find the sum of the first 18 terms of 2, –8, 32, –128, …

|  |  |
| --- | --- |
| *Sn* = | Use the formula for the sum of the first *n* terms of a geometric sequence. |
| *S*18 = | *a*1 = 2, *r* = –4, and *n* = 18; this formula represents the sum of the first 18 terms. |
| = –27,487,790,694 | Use a calculator. |

The sum of the first 18 terms is –27,487,790,694. Equivalently, this sum is the 18th partial sum of the sequence 2, –8, 32, –128, ….

**Example III.C.2:** Find 

**Solution:**

Write out a few terms in the sum.

= 6 ∙ 21 + 6 ∙ 22 + 6 ∙ 23 + 6 ∙ 24 + … + 6 ∙ 210

Notice that each term after the first is obtained by multiplying the preceding term by 2. To find the sum of the 10 terms (*n* = 10), *a*1 and *r* must be known. The first term, *a*1, is 6 · 2, or 12. The common ratio, *r*, is 2.

|  |  |
| --- | --- |
| *Sn* = | Use the formula for the sum of the first *n* terms of a geometric sequence. |
| *S*10 = | *a*1 (the first term) = 12, *r* = 2, and *r* = 10; this formula represents the sum of the first 10 terms. |
| = 12,276 | Use a calculator. |

Thus,

= 12,276

**Example III.C.3:** A union contract specifies that each worker will receive a five-percent pay increase each year for the next 30 years. One worker is paid $20,000 the first year. What is this person's total salary over a 30-year period?

**Solution:**

The salary for the first year, *a*1, is $20,000. With a five-percent raise, the second-year salary is computer as follows:

*a*2 = 20,000 + 20,000(0.05) = 20,000(1.05)

Each year, the employee's salary is 1.05 times what it was in the previous year. Thus, the salary for year 3 is 1.05 times 20,000(1.05), or 20,000(1.05)2. The worker's salary for each of the first five years is given in the table below:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Yearly Salary | | | | | |
| **Year 1** | **Year 2** | **Year 3** | **Year 4** | **Year 5** | **…** |
| 20,000 | 20,000(1.05)1 | 20,000(1.05)2 | 20,000(1.05)3 | 20,000(1.05)4 |  |

The numbers in the bottom row form a geometric sequence with *a*1 = 20,000 and *r* = 1.05. To find the total salary over 30 years, use the formula for the sum of the first *n* terms of a geometric sequence, with *n* = 30.

|  |  |
| --- | --- |
| *Sn* = |  |
| *S*30 = |  |
| = |  |
| ≈ $1,328,777 | Use a calculator. |

The total salary over the 30-year period is approximately $1,328,777.

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